Continuous symmetries of the lattice potential KdV equation

2007 J. Phys. A: Math. Theor. 40 4141

(http://iopscience.iop.org/1751-8121/40/15/006)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 4141–4159 doi:10.1088/1751-8113/40/15/006

Continuous symmetries of the lattice potential KdV equation

Decio Levi[1] **and Matteo Petrera**[2,3]

[1] Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
[2] Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D-85747 Garching bei München, Deutschland
[3] Dipartimento di Fisica E. Amaldi, Università degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy

E-mail: levi@fis.uniroma3.it and petrera@ma.tum.de

Received 31 January 2007
Published 23 March 2007
Online at stacks.iop.org/JPhysA/40/4141

Abstract
In this paper we present a set of results on the integration and on the symmetries of the lattice potential Korteweg–de Vries (lpKdV) equation. Using its associated spectral problem we construct the soliton solutions and the Lax technique enables us to provide infinite sequences of generalized symmetries. Finally, using a discrete symmetry of the lpKdV equation, we construct a large class of non-autonomous generalized symmetries.

PACS numbers: 02.30.Ik, 02.30.Jr

1. Introduction

The lattice version of the potential Korteweg–de Vries (lpKdV) equation

$$w_t = w_{xxx} + 3w_x^2 \tag{1}$$

is given by the nonlinear partial difference equation [17]:

$$\mathbb{D} \doteq (p - q + u_{n,m+1} - u_{n+1,m})(p + q - u_{n+1,m+1} + u_{n,m}) - (p^2 - q^2) = 0. \tag{2}$$

The above equation is probably the best known completely discrete equation which involves just four points which lay on two orthogonal infinite lattices and are the vertices of an elementary square—a quad-graph—(see figure 1). Equation (2) is one of the lattice equations on quad-graphs classified in [1], where the $3D$ *consistency* is used as a tool to establish its integrability.

The lpKdV equation has been introduced for the first time by Hirota [8] in 1977 and it is nothing else but the nonlinear superposition formula for the Korteweg–de Vries equation under disguise. A review of results about the lpKdV equation can be found in [17].

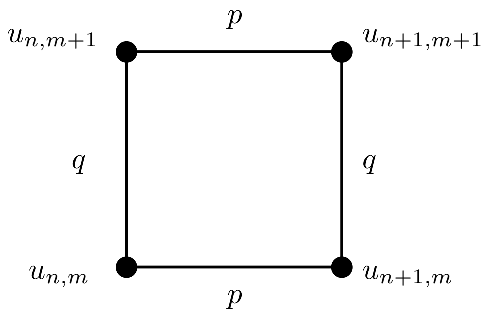

Figure 1. An elementary square.

Integrable equations possess an infinite set of symmetries. Few of them are *point symmetries*, i.e. symmetries whose infinitesimal generators depend just on the independent and dependent variables, while an infinite denumerable number of them are *generalized symmetries*. The latter ones depend on the derivatives of the dependent variable with respect to the continuous independent variables and on a few lattice points if the independent variables are discrete. The presence of this infinite Lie algebra of symmetries and of the associated conserved quantities is one of the most important features of the integrability of a given nonlinear equation and it has been used with profit in the past to provide integrability tests for several partial differential equations in $\mathbb{R}^2$ and differential-difference equations [15, 24].

To be able to introduce an integrability test based on symmetry one needs to understand the structure of the infinite dimensional symmetry algebra of integrable equations. In the case of completely discrete equations the situation is not as clear as for the differential-difference or the partial-differential case. A result in this direction has been obtained for the discrete-time Toda lattice [11]. However, the Toda lattice is just an example and more examples are needed to get a sufficiently general idea of the possible structures which may appear.

On the other hand, in [9, 10, 12], the multiscale expansion technique [22] has been extended to the case of equations defined on a lattice. In [10] one of the authors has performed, as an example, the multiscale expansion of equation (2) deriving a completely local discrete nonlinear Schrödinger equation (dNLS). The integrability of this dNLS equation has been questioned by many people. So the construction of the symmetries of the lpKdV equation and their multiscale expansion should be a concrete tool to check its integrability.

The present paper is devoted to the study of the lpKdV equation exactly with these aims. In section 2 we review some known results on this lattice equation, while in section 3 we present its inverse scattering transform, which turns out to be a slight generalization of the results presented by Boiti and collaborators on an asymmetric discretization of the continuous Schrödinger spectral problem [3]. Section 4 is devoted to the construction of Lie point and generalized symmetries. Finally, section 5 contains some concluding remarks and some open problems.

2. The lattice potential KdV equation

Here we present some known results on the lpKdV equation (2) and its integrability.

In equation (2) $u_{n,m}$ is the dynamical field variable, which we assume to be real and asymptotically bounded by a constant, defined at the site $(m, n) \in \mathbb{Z}^2$ while p, q are two nonzero real parameters which are related to the lattice steps and will go to infinity in the continuous limit so as to obtain from equation (2) the continuous potential KdV equation (1).

Since we have two discrete independent variables, n and m, we perform the continuous limit in two steps. Each step is achieved by shrinking the corresponding lattice step to zero and sending to infinity the number of lattice points.

In the first step we define $u_{n,m} \doteq v_k(\tau)$, where $k \doteq n+m$ and $\tau \doteq \delta m$, being $\delta \doteq p-q$ the lattice step in the m-direction. In the limit $m \to \infty$, $\delta \to 0$, we obtain the differential-difference equation

$$\frac{\partial v_k}{\partial \tau} = \frac{2p}{2p - v_{k+1} + v_{k-1}} - 1. \tag{3}$$

Defining

$$q_k \doteq 2p - v_{k+2} + v_k,$$

we can rewrite equation (3) as the differential-difference equation

$$\frac{\partial q_k}{\partial \tau} = 2p\left(\frac{1}{q_{k-1}} - \frac{1}{q_{k+1}}\right). \tag{4}$$

Equation (4) has been obtained in [2] as the simplest local evolution equation associated with the asymmetric discrete Schrödinger spectral problem

$$\psi_{k+2} = q_k\psi_{k+1} + \lambda\psi_k, \tag{5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter.

Introducing the new field $s_k \doteq (2p)/q_k$, equation (4) reads

$$\frac{\partial s_k}{\partial \tau} = s_k^2(s_{k+1} - s_{k-1}), \tag{6}$$

the so-called *discrete KdV equation* [16]. Then the Miura transformation, $a_k \doteq s_k s_{k-1}$, maps equation (6) into the *discrete Volterra equation*

$$\frac{\partial a_k}{\partial \tau} = a_k(a_{k+1} - a_{k-1}), \tag{7}$$

associated with the Schrödinger spectral problem

$$\psi_{k-1} + a_k\psi_{k+1} = \mu\psi_k,$$

where $\mu \in \mathbb{C}$ is a spectral parameter.

The second continuous limit of equation (2) is performed by defining $v_k(\tau) \doteq w(x,t)$, with $x \doteq 2(k+\tau/p)/p$ and $t \doteq 2(k/3+\tau/p)/p^3$. If we carry out the limit $p \to \infty$, $k \to \infty$, $\tau \to \infty$, in such a way that x and t remain finite, then equation (3) is transformed into the potential KdV equation (1).

The integrability of the lpKdV equation (2) is proven in [17] by giving its Lax pair, an overdetermined system of matrix equations for the vector $\Psi^h_{n,m} \doteq \left(\psi^1_{n,m}(h), \psi^2_{n,m}(h)\right)^T$, where $h \in \mathbb{C}$ is a spectral parameter:

$$\Psi^h_{n+1,m} = L^h_{n,m}\Psi^h_{n,m}, \tag{8a}$$
$$\Psi^h_{n,m+1} = M^h_{n,m}\Psi^h_{n,m}, \tag{8b}$$

with

$$L^h_{n,m} \doteq \begin{pmatrix} p - u_{n+1,m} & 1 \\ h^2 - p^2 + (p+u_{n,m})(p-u_{n+1,m}) & p + u_{n,m} \end{pmatrix},$$

and

$$M^h_{n,m} \doteq \begin{pmatrix} q - u_{n,m+1} & 1 \\ h^2 - q^2 + (q + u_{n,m})(q - u_{n,m+1}) & q + u_{n,m} \end{pmatrix}.$$

The consistency of equations (8) implies the discrete Lax equation

$$L^h_{n,m+1} M^h_{n,m} = M^h_{n+1,m} L^h_{n,m},$$

which plays the same role as the zero-curvature equation $L_{,t} = [M, L]$ in the continuous case. The lpKdV equation (2) corresponds to an isospectral deformation of the Lax pair (8), i.e. whenever h is an m and n-independent complex constant. We can rewrite equations (8) in scalar form in terms of $\psi_{n,m} \doteq \psi^1_{n,m}(h)$:

$$\psi_{n+2,m} = (2p - u_{n+2,m} + u_{n,m})\psi_{n+1,m} + (h^2 - p^2)\psi_{n,m}, \tag{9a}$$

$$\psi_{n,m+1} = \psi_{n+1,m} + (q - p + u_{n+1,m} - u_{n,m+1})\psi_{n,m}. \tag{9b}$$

Taking into account equations (2)–(9b) we can rewrite the m-evolution as

$$\psi_{n,m+2} = (2q - u_{n,m+2} + u_{n,m})\psi_{n,m+1} + (h^2 - q^2)\psi_{n,m}. \tag{10}$$

Equation (10) coincides with equation (9a) swapping the indices $n \leftrightarrow m$ and $p \leftrightarrow q$. It is easy to see that also the lpKdV equation (2) has the same discrete symmetry.

Defining

$$q_n \doteq q_{n,m} \doteq 2p - u_{n+2,m} + u_{n,m}, \tag{11}$$

choosing $\lambda \doteq h^2 - p^2$ and dropping the parametric dependence on m we see that equation (9a) is equivalent to the discrete Schrödinger spectral problem (5). The m-evolution, provided by equation (9b), cannot be written in a simple way in terms of $q_{n,m}$.

3. The discrete spectral problem associated with the lpKdV equation

Let us now study the direct and inverse problems associated with equation (5). Our results are a generalization of those contained in [3]: in fact they reduce to them when $p = 1$. Henceforth we remand to [3] for most of the technical details.

As the independent discrete variable m—the discrete time variable of the lpKdV equation—enters parametrically in the results of this section, it will not be explicitly written. We shall write it just when it is necessary, namely when we will study the m-evolution of the spectral data associated with the lpKdV equation.

3.1. Direct problem

The solutions $u_{n,m}$ of equation (2) must go asymptotically to an arbitrary constant to be consistent with the difference equation (2). Then q_n, defined in equation (11), goes asymptotically to $2p$.

Following [3] we define

$$h \doteq \mathrm{i}\kappa, \qquad \lambda \doteq -\kappa^2 - p^2, \tag{12a}$$

$$q_n \doteq \eta_n + 2p, \qquad \psi_n \doteq (p + \mathrm{i}\kappa)^n \chi_n, \tag{12b}$$

where $\eta_n \doteq -u_{n+2,m} + u_{n,m}$ is a new asymptotically vanishing potential. In terms of the new variables (12) the spectral problem (5) reads

$$(p + \mathrm{i}\kappa)\chi_{n+2} - 2p\chi_{n+1} + (p - \mathrm{i}\kappa)\chi_n = \eta_n \chi_{n+1}. \tag{13}$$

The Jost solutions $\mu_n^{\pm}$ of the spectral problem (13) are defined in terms of the potential η_n and of the discrete complex exponential $[(p+\mathrm{i}\kappa)/(p-\mathrm{i}\kappa)]^n$ through the following 'discrete integral equations'

$$\mu_n^{+} = 1 - \frac{1}{2\mathrm{i}k}\sum_{j=n+1}^{+\infty}\left[1+\left(\frac{p+\mathrm{i}\kappa}{p-\mathrm{i}\kappa}\right)^{j-n}\right]\eta_{j-1}\mu_j^{+}, \tag{14a}$$

$$\mu_n^{-} = 1 + \frac{1}{2\mathrm{i}\kappa}\sum_{j=-\infty}^{n}\left[1+\left(\frac{p+\mathrm{i}\kappa}{p-\mathrm{i}\kappa}\right)^{j-n}\right]\eta_{j-1}\mu_j^{-}. \tag{14b}$$

The Jost solution μ_n^{+} is an analytic function of κ for $\mathrm{Im}(\kappa) > 0$ and μ_n^{-} for $\mathrm{Im}(\kappa) < 0$ with the boundary conditions

$$\lim_{n\to+\infty}\mu_n^{+} = 1, \qquad \mathrm{Im}(\kappa) \geqslant 0, \tag{15a}$$

$$\lim_{n\to-\infty}\mu_n^{-} = 1, \qquad \mathrm{Im}(\kappa) \leqslant 0. \tag{15b}$$

For $\mathrm{Im}(\kappa) = 0$ we define the spectral data:

$$a^{\pm}(\kappa) \doteq 1 \mp \frac{1}{2\mathrm{i}\kappa}\sum_{j=-\infty}^{+\infty}\eta_{j-1}\mu_j^{\pm},$$

$$b^{\pm}(\kappa) \doteq \pm\frac{1}{2\mathrm{i}\kappa}\sum_{j=-\infty}^{+\infty}\left(\frac{p+\mathrm{i}\kappa}{p-\mathrm{i}\kappa}\right)^{j}\eta_{j-1}\mu_j^{\pm}.$$

Due to the analyticity property of the Jost solutions one can prove that $a^{+}(k)$ can be analytically extended to $\mathrm{Im}(\kappa) > 0$ and $a^{-}(\kappa)$ to $\mathrm{Im}(\kappa) < 0$. The inverse of the functions $a^{\pm}(\kappa)$, namely $T^{\pm}(\kappa) \doteq [a^{\pm}(\kappa)]^{-1}$, play the role of the *transmission coefficients* and the functions $R^{\pm}(\kappa) \doteq b^{\pm}(\kappa)/a^{\pm}(\kappa)$ are the *reflection coefficients*. The poles of $T^{\pm}(\kappa)$ are related to the soliton solutions of the evolution equations associated with the spectral problem (13). Assuming that equations (14) are solvable and their solutions are unique, we obtain for $\mathrm{Im}(\kappa) = 0$ the following relations between the Jost solutions and the spectral data,

$$\mu_n^{\pm}(\kappa) = a^{\pm}(\kappa)\mu_n^{\mp}(\kappa) + \left(\frac{p-\mathrm{i}\kappa}{p+\mathrm{i}\kappa}\right)^{n} b^{\pm}(\kappa)\mu_n^{\mp}(-\kappa). \tag{16}$$

Taking into account the asymptotic limits (15) of the Jost functions, for $\mathrm{Im}(\kappa) = 0$ we get

$$\mu_n^{\pm} \sim [T^{\pm}(\kappa)]^{-1}\left[1+\left(\frac{p-\mathrm{i}\kappa}{p+\mathrm{i}\kappa}\right)^{n}R^{\pm}(\kappa)\right], \qquad n \to \mp\infty. \tag{17}$$

Let us assume that the function $a^{+}(\kappa)$ has N simple zeros at $\kappa = \kappa_j^{+}, 1 \leqslant j \leqslant N$. In correspondence with these values of κ the Jost function is asymptotically bounded and the N residues of the transmission coefficient $T^{+}(\kappa)$, say $\left\{c_j^{+}\right\}_{j=1}^{N}$, are associated with its normalization coefficient.

Henceforth we can associate with a given potential η_n in a unique way the spectral data $\mathsf{S}[\eta_n] \doteq \left\{R(\kappa), T(\kappa), \{c_j\}_{j=1}^{N}\right\}$, obtained as a solution of the spectral problem (13). As the spectral problem (13) is a linear ordinary difference equation in n, the solution χ_n is defined up to an arbitrary normalization function $\Omega_m(\kappa)$.

3.2. Inverse problem

Taking into account equations (15)–(17) and using the Cauchy–Green formula we can write the following integral equations for the two Jost solutions

$$T(\kappa)\mu_n^+(\kappa) = 1 + \sum_{j=1}^{N} C_j \frac{\mu_n^-(-\kappa_j^+)}{\kappa - \kappa_j^+} \left(\frac{p - \mathrm{i}\kappa_j^+}{p + \mathrm{i}\kappa_j^+}\right)^n + \\ + \frac{1}{2\mathrm{i}\kappa} \int_{-\infty}^{+\infty} \frac{\mu_n^-(-s)R(s)}{s - \kappa - \mathrm{i}0} \left(\frac{p - \mathrm{i}s}{p + \mathrm{i}s}\right)^n \mathrm{d}s, \qquad \mathrm{Im}(\kappa) \leqslant 0, \tag{18a}$$

$$\mu_n^-(\kappa) = 1 + \sum_{j=1}^{N} C_j \frac{\mu_n^-(-\kappa_j^+)}{\kappa - \kappa_j^+} \left(\frac{p - \mathrm{i}\kappa_j^+}{p + \mathrm{i}\kappa_j^+}\right)^n \\ + \frac{1}{2\mathrm{i}\kappa} \int_{-\infty}^{+\infty} \frac{\mu_n^-(-s)R(s)}{s - \kappa + \mathrm{i}0} \left(\frac{p - \mathrm{i}s}{p + \mathrm{i}s}\right)^n \mathrm{d}s, \qquad \mathrm{Im}(\kappa) \geqslant 0, \tag{18b}$$

where

$$C_j \doteq \frac{c_j^+}{2\mathrm{i}\kappa_j^+} \sum_{k=-\infty}^{+\infty} \left(\frac{p + \mathrm{i}\kappa_j^+}{p - \mathrm{i}\kappa_j^+}\right)^k \eta_{k-1}\mu_k^+(\kappa_j^+). \tag{19}$$

Therefore, given the spectral data $\mathsf{S}[\eta_n]$, we are able to reconstruct the Jost solutions and, in terms of them, the potential

$$\eta_{n,m} = \mathrm{i} \sum_{j=1}^{N} C_j \left[\mu_{n+2,m}^-(-\kappa_j^+) \left(\frac{p + \mathrm{i}\kappa_j^+}{p - \mathrm{i}\kappa_j^+}\right)^2 - \mu_{n,m}^-(-\kappa_j^+)\right] \left(\frac{p + \mathrm{i}\kappa_j^+}{p - \mathrm{i}\kappa_j^+}\right)^n \\ - \int_{-\infty}^{+\infty} \left[\mu_{n+2,m}^-(-s) \left(\frac{p + \mathrm{i}s}{p - \mathrm{i}s}\right)^2 - \mu_{n,m}^-(-s)\right] \left(\frac{p + \mathrm{i}s}{p - \mathrm{i}s}\right)^n R_m^+(s)\,\mathrm{d}s. \tag{20}$$

We remand to [3] for further details and for a study of the convergence of the sum appearing in the definition (19) of the coefficients C_j.

3.3. Time evolution of spectral data

Taking into account the definitions (12), the m-evolution of $\chi_{n,m}$ is obtained from equation (9*b*) and is given by

$$\chi_{n,m+1} = (p + \mathrm{i}\kappa)\chi_{n+1,m} + (q - p + u_{n+1,m} - u_{n,m+1})\chi_{n,m}. \tag{21}$$

As the lpKdV equation (2) is an isospectral deformation of equation (13), from (12*a*) κ is an m-independent parameter.

Defining $\chi_{n,m} \doteq \Omega_m(\kappa)\mu_{n,m}^+$, where $\Omega_m(\kappa)$ is a normalization function, and taking into account the asymptotic behaviours (15)–(17) of the Jost functions, we find from equation (21) the following time evolution of $\mathsf{S}[\eta_{n,m}]$:

$$T_{m+1}(\kappa) = T_m(\kappa), \qquad R_{m+1}(\kappa) = \left(\frac{q - \mathrm{i}\kappa}{q + \mathrm{i}\kappa}\right) R_m(\kappa). \tag{22}$$

Integrating equations (22) we get

$$T_m(\kappa) = T(\kappa), \qquad R_m(\kappa) = \left(\frac{q - \mathrm{i}\kappa}{q + \mathrm{i}\kappa}\right)^m R(\kappa),$$

namely the transmission coefficient is invariant under the evolution of the lpKdV equation (2), while the reflection coefficient acquires an m-dependent discrete exponential factor.

3.4. Soliton solutions

The soliton solutions of equation (2) are obtained by considering the reflectionless solution of the spectral problem (9*a*)–(9*b*) and (12). They are obtained from equation (20) setting $R_m^+(\kappa) = 0$. The Jost solutions $\mu_{n,m}^-\left(-\mathrm{i}\kappa_j^+\right)$ are obtained solving equation (18*b*) for $\kappa = -\kappa_j^+$. For $N = 1$ they are given by

$$\mu_{n,m}^-(-\kappa^+) = \left[1 + \frac{C_m}{2\kappa^+}\left(\frac{p - \mathrm{i}\kappa^+}{p + \mathrm{i}\kappa^+}\right)^n\right]^{-1},$$

while for $N = 2$ they read

$$\mu_{n,m}^-\left(-\kappa_1^+\right) = \frac{1}{\mathcal{D}_{1,2}}\left[1 + \frac{C_{2,m}}{\kappa_2^+ - \kappa_1^+}\left(\frac{p - \mathrm{i}\kappa_2^+}{p + \mathrm{i}\kappa_2^+}\right)^n\right],$$

$$\mu_{n,m}^-\left(-\kappa_2^+\right) = \frac{1}{\mathcal{D}_{1,2}}\left[1 + \frac{C_{1,m}}{\kappa_1^+ - \kappa_2^+}\left(\frac{p - \mathrm{i}\kappa_1^+}{p + \mathrm{i}\kappa_1^+}\right)^n\right],$$

where

$$\mathcal{D}_{1,2} \doteq 1 + \frac{C_{2,m}}{2\kappa_2^+}\left(\frac{p - \mathrm{i}\kappa_2^+}{p + \mathrm{i}\kappa_2^+}\right)^n + \frac{C_{1,m}}{2\kappa_1^+}\left(\frac{p - \mathrm{i}\kappa_1^+}{p + \mathrm{i}\kappa_1^+}\right)^n + \frac{C_{1,m}C_{2,m}\left(\kappa_2^+ - \kappa_1^+\right)^2}{4\kappa_1^+\kappa_2^+\left(\kappa_2^+ + \kappa_1^+\right)^2}\left(\frac{p - \mathrm{i}\kappa_1^+}{p + \mathrm{i}\kappa_1^+}\right)^n\left(\frac{p - \mathrm{i}\kappa_2^+}{p + \mathrm{i}\kappa_2^+}\right)^n.$$

Taking into account the relation between $\eta_{n,m}$ and $u_{n,m}$ and the time evolution (22) we get that the one soliton solution for the lpKdV equation (2) is given by

$$u_{n,m} = -\frac{\mathrm{i}C\left(\frac{p-\mathrm{i}\kappa^+}{p+\mathrm{i}\kappa^+}\right)^n\left(\frac{q-\mathrm{i}\kappa^+}{q+\mathrm{i}\kappa^+}\right)^m}{1 + \frac{C}{2\kappa^+}\left(\frac{p-\mathrm{i}\kappa^+}{p+\mathrm{i}\kappa^+}\right)^n\left(\frac{q-\mathrm{i}\kappa^+}{q+\mathrm{i}\kappa^+}\right)^m}. \tag{23}$$

By proper choices of the complex constants C and κ^+, where κ^+ is defined in the upper half plane, and of the real constants p and q one can always render the solution (23) real.

The two soliton solution is given by

$$u_{n,m} = -\frac{\mathrm{i}}{\mathcal{D}_{1,2}}\sum_{j=1}^{2}\frac{C_j\left(\frac{p-\mathrm{i}\kappa_j^+}{p+\mathrm{i}\kappa_j^+}\right)^n\left(\frac{q-\mathrm{i}\kappa_j^+}{q+\mathrm{i}\kappa_j^+}\right)^m}{1 + \frac{C_j}{2\kappa_j^+}\left(\frac{p-\mathrm{i}\kappa_j^+}{p+\mathrm{i}\kappa_j^+}\right)^n\left(\frac{q-\mathrm{i}\kappa_j^+}{q+\mathrm{i}\kappa_j^+}\right)^m},$$

with $\mathcal{D}_{1,2}$ given above.

4. Continuous symmetries of the lpKdV equation

Lie symmetries of the lpKdV equation (2) are given by those continuous transformations which leave the equation invariant. From the infinitesimal point of view they are obtained by requiring the infinitesimal invariant condition

$$\mathrm{pr}\,\widehat{X}_{n,m}\mathbb{D}|_{\mathbb{D}=0} = 0, \tag{24}$$

where

$$\widehat{X}_{n,m} = F_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots)\partial_{u_{n,m}}. \tag{25}$$

By $\mathrm{pr}\,\widehat{X}_{n,m}$ we mean the prolongation of the infinitesimal generator $\widehat{X}_{n,m}$ to all points appearing in $\mathbb{D} = 0$.

If $F_{n,m} = F_{n,m}(u_{n,m})$ then we get *point symmetries* and the procedure to get them from equation (24) is purely algorithmic [14]. *Generalized symmetry* is obtained when

$F_{n,m} = F_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots)$. In the case of nonlinear discrete equations, the Lie point symmetries are not very common, but, if the equation is integrable and there exists a Lax pair, it is possible to construct an infinite family of generalized symmetries.

In correspondence with the infinitesimal generator (25) we can in principle construct a group transformation by integrating the initial boundary problem

$$\frac{\mathrm{d}\widetilde{u}_{n,m}(\varepsilon)}{\mathrm{d}\varepsilon} = F_{n,m}(\widetilde{u}_{n,m}(\varepsilon), \widetilde{u}_{n\pm1,m}(\varepsilon), \widetilde{u}_{n,m\pm1}(\varepsilon), \ldots), \tag{26}$$

with $\widetilde{u}_{n,m}(\varepsilon = 0) = u_{n,m}$, where $\varepsilon \in \mathbb{R}$ is the continuous Lie group parameter. This can be done effectively only in the case of point symmetries as in the generalized case we have a differential-difference equation for which we cannot find the solution for a generic initial data, but, at most, we can find some particular solutions. Equation (24) is equivalent to the request that the ε-derivative of the equation $\mathbb{D} = 0$, written for $\widetilde{u}_{n,m}(\varepsilon)$, is identically satisfied when the ε-evolution of $\widetilde{u}_{n,m}(\varepsilon)$ is given by equation (26). This is also equivalent to say that the flows (in the group parameter space) given by equation (26) are compatible or commute with $\mathbb{D} = 0$.

By means of a symmetry transformation we can construct an ε-dependent class of solutions of equation (2), given by the functions $\widetilde{u}_{n,m}(\varepsilon)$. We can associate with this class of solutions the corresponding solution $\widetilde{\psi}_n$ of the spectral problem (9*a*) and equation (26) can be found among those nonlinear evolution equations associated with the spectral problem (9*a*) which are commuting with equation (2).

4.1. Lie point symmetries

As we do not want to change the basic lattice structure of our space, we just consider transformations G_ε acting on the domain of the dependent variables:

$$G_\varepsilon : u_{n,m} \mapsto \Phi_{n,m}(u_{n,m}; \varepsilon), \qquad \varepsilon \in \mathbb{R}.$$

The infinitesimal generator of the group action of G_ε on $u_{n,m}$ is the vector field

$$\widehat{X}_{n,m} = \phi_{n,m}(u_{n,m})\partial_{u_{n,m}},$$

where

$$\phi_{n,m}(u_{n,m}) \doteq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi_{n,m}(u_{n,m}; \varepsilon)\bigg|_{\varepsilon=0}.$$

There is a one-to-one correspondence between connected groups of transformations and their associated infinitesimal generators [20]. The group action of G_ε can be reconstructed from the flow of the infinitesimal vector field $\widehat{X}_{n,m}$ either by exponentiation:

$$\Phi_{n,m}(u_{n,m}; \varepsilon) = \exp(\varepsilon\widehat{X}_{n,m})u_{n,m},$$

or by solving the differential boundary value problem

$$\frac{\mathrm{d}\widetilde{u}_{n,m}(\varepsilon)}{\mathrm{d}\varepsilon} = \phi_{n,m}(\widetilde{u}_{n,m}(\varepsilon)), \tag{27}$$

with the initial condition $\widetilde{u}_{n,m}(\varepsilon = 0) = u_{n,m}$. Equation (27) is just a subcase of equation (26) when $F_{n,m}$ depends just on $u_{n,m}$. The prolongation of the infinitesimal action of G_ε is given by

$$\begin{aligned} \mathrm{pr}\,\widehat{X}_{n,m} = {} & \phi_{n,m}(u_{n,m})\partial_{u_{n,m}} + \phi_{n+1,m}(u_{n+1,m})\partial_{u_{n+1,m}} \\ & + \phi_{n,m+1}(u_{n,m+1})\partial_{u_{n,m+1}} + \phi_{n+1,m+1}(u_{n+1,m+1})\partial_{u_{n+1,m+1}}. \end{aligned} \tag{28}$$

Applying the prolongation (28) to equation (2) we get

$$\operatorname{pr}\widehat{X}_{n,m}\mathbb{D}|_{\mathbb{D}=0} = [\phi_{n,m}(u_{n,m}) - \phi_{n+1,m+1}(u_{n+1,m+1})](p - q + u_{n,m+1} - u_{n+1,m}) \\ + (p + q - u_{n+1,m+1} + u_{n,m})[\phi_{n,m+1}(u_{n,m+1}) - \phi_{n+1,m}(u_{n+1,m})]|_{\mathbb{D}=0} = 0. \quad (29)$$

Due to equation (2) just three of the four different fields appearing in equation (29) are independent and, without loss of generality, we choose them to be $u_{n,m}$, $u_{n,m+1}$ and $u_{n+1,m}$.

From equation (29), by differentiation with respect to $u_{n,m}$, we obtain, when $p \neq q$,

$$\frac{\mathrm{d}\phi_{n,m}}{\mathrm{d}u_{n,m}} = \frac{\mathrm{d}\phi_{n+1,m+1}}{\mathrm{d}u_{n+1,m+1}}, \quad (30)$$

so that

$$\frac{\mathrm{d}^2\phi_{n,m}}{\mathrm{d}u_{n,m}^2} = 0 \qquad \Rightarrow \qquad \phi_{n,m} = \mathcal{K}^0_{n,m} + \mathcal{K}^1_{n,m}u_{n,m}, \quad (31)$$

where $\mathcal{K}^0_{n,m}$ and $\mathcal{K}^1_{n,m}$ are two integration constants depending just upon n and m. Inserting $\phi_{n,m}$ given in equation (31) into equation (30) we find that $\mathcal{K}^1_{n,m} = \mathcal{K}^1_{n+1,m+1}$, while if we insert it into equation (29) we get an explicit equation in $u_{n,m}$, $u_{n,m+1}$ and $u_{n+1,m}$. Then the various powers of $u_{n,m}$, $u_{n,m+1}$ and $u_{n+1,m}$ give us a set of coupled linear partial difference equations for $\mathcal{K}^0_{n,m}$ and $\mathcal{K}^1_{n,m}$, whose solutions are

$$\begin{aligned} \mathcal{K}^0_{n,m} &= \gamma_1 + (-1)^{n+m}[\gamma_2 - (qm + pn)\gamma_3], \\ \mathcal{K}^1_{n,m} &= (-1)^{n+m}\gamma_3, \end{aligned} \quad (32)$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary constants. Therefore the lpKdV equation (2) admits a three-dimensional group G_ε of Lie point symmetries, whose infinitesimal generators are

$$\widehat{X}^1_{n,m} = \partial_{u_{n,m}}, \quad (33a)$$
$$\widehat{X}^2_{n,m} = (-1)^{n+m}\partial_{u_{n,m}}, \quad (33b)$$
$$\widehat{X}^3_{n,m} = (-1)^{n+m}[u_{n,m} - (pn + qm)]\partial_{u_{n,m}}. \quad (33c)$$

The commutation relations between the generators $\widehat{X}^1_{n,m}, \widehat{X}^2_{n,m}, \widehat{X}^3_{n,m}$ are

$$\left[\widehat{X}^1_{n,m}, \widehat{X}^2_{n,m}\right] = 0, \qquad \left[\widehat{X}^1_{n,m}, \widehat{X}^3_{n,m}\right] = \widehat{X}^2_{n,m}, \qquad \left[\widehat{X}^2_{n,m}, \widehat{X}^3_{n,m}\right] = \widehat{X}^1_{n,m},$$

so that they span a solvable Lie algebra.

The finite transformation generated by the symmetry generators (33) is given by

$$\widetilde{u}_{n,m} = \mathrm{e}^{\varepsilon_3(-1)^{n+m}}u_{n,m} + \frac{\mathrm{e}^{\varepsilon_3(-1)^{n+m}} - 1}{\varepsilon_3(-1)^{n+m}}[\varepsilon_1 + \varepsilon_2(-1)^{n+m} - \varepsilon_3(-1)^{n+m}(pn + qm)], \quad (34)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the group parameters associated respectively with the symmetry generators $\widehat{X}^1_{n,m}, \widehat{X}^2_{n,m}, \widehat{X}^3_{n,m}$.

When $p = q$ equation (2) simplifies to the product of two linear discrete wave equations:

$$(u_{n,m} - u_{n+1,m+1} + 2p)(u_{n+1,m} - u_{n,m+1}) = 0,$$

and the symmetry group becomes infinite dimensional.

Finally, let us recall that equation (2) admits also some *discrete symmetries*:

(1) The exchange of n and m together with the exchange of p and q leaves the lpKdV equation invariant.
(2) The exchange of n and $n + 1$ together with the transformation of p into $-p$ leaves the lpKdV equation invariant.
(3) The exchange of m and $m + 1$ together with the transformation of q into $-q$ leaves the lpKdV equation invariant.

4.2. Generalized symmetries

A generalized symmetry is obtained when the function $F_{n,m}$ appearing in equation (26) depends effectively on $u_{n,m}$ evaluated in some lattice points. A way to obtain it is to look at those differential-difference equations (26) associated with the spectral problem (9*a*) which are compatible with equation (2). If the nonlinear differential-difference equation (26) is associated with a spectral problem, the proof that it is a symmetry of equation (26) can be greatly simplified. In fact in this case the spectral problem establishes a one-to-one correspondence between equation (26) and the spectral data and thus one can prove the commutativity of the flows in the space of the spectral data, where the equations are linear.

With this aim, in the following, we construct the hierarchies of nonlinear equations associated with the spectral problem (9*a*) of the lpKdV equation (2). To do so, let us rewrite equation (9*a*) as an operator equation

$$L_n\psi_n = \lambda\psi_n, \qquad L_n \doteq E_n^2 - (\eta_{n,m} + 2p)E_n, \tag{35}$$

where $E_n^{\pm s}\psi_n \doteq \psi_{n\pm s}$, $s \in \mathbb{N}$, are the finite shift operators in the n variable the eigenvalue $\lambda \in \mathbb{C}$ is defined in equation (12*a*) and $\eta_{n,m} \doteq u_{n,m} - u_{n+2,m}$ is an asymptotically vanishing potential. Starting from equation (35) we can apply the *Lax technique*, to obtain the recursion operator $\mathcal{L}_n$ and the hierarchy of nonlinear evolution equations associated with it, as it has been done in [2]. By the Lax technique we mean that constructive procedure introduced by Bruschi and Ragnisco [4] in 1980 which consists in finding all those operators M_n which are compatible with equation (35) such that $\mathrm{d}\psi_n/\mathrm{d}\varepsilon = -M_n\psi_n$. According to the dependence of λ on the group parameter ε we can distinguish between two different operator equations for the compatible L_n and M_n:

(1) if $\mathrm{d}\lambda/\mathrm{d}\varepsilon = 0$, then

$$\frac{\mathrm{d}L_n}{\mathrm{d}\varepsilon} = [L_n, M_n], \qquad \text{(isospectral hierarchy)}; \tag{36}$$

(2) if $\mathrm{d}\lambda/\mathrm{d}\varepsilon \neq 0$, i.e. $\mathrm{d}\lambda/\mathrm{d}\varepsilon = f(\lambda, \varepsilon)$, then

$$\frac{\mathrm{d}L_n}{\mathrm{d}\varepsilon} = [L_n, M_n] + f(L_n, \varepsilon), \qquad \text{(non-isospectral hierarchy)}. \tag{37}$$

Each one of the two cases will provide symmetries for the lpKdV equation (2).

4.2.1. The isospectral hierarchy. The isospectral hierarchy of equations is obtained looking at those differential-difference equations whose Lax equation is given by equation (36). As we are considering differential-difference equations, M_n is a suitable operator expressed in terms of shift operators $\{E_n^{\pm s}\}$. Taking into account equation (35), the left-hand side of equation (36) is given by $-(\mathrm{d}\eta_{n,m}(\varepsilon)/\mathrm{d}\varepsilon)E_n$, i.e. it is an operator of the form $V_n(\varepsilon)E_n$, where $V_n(\varepsilon)$ is a scalar function. The Lax technique described in [4] allows us to write down the following relation:

$$\widetilde{V}_n = \mathcal{L}_n V_n + V_n^{(0)}, \tag{38}$$

where $\widetilde{V}_n = -(\mathrm{d}\eta_{n,m}(\widetilde{\varepsilon})/\mathrm{d}\widetilde{\varepsilon})$, $V_n^{(0)}$ is a given function of $\eta_{n,m}$ and of a certain number of arbitrary constants. In this case $V_n^{(0)}$ reads [2]

$$V_n^{(0)} \doteq q_{n,m}\left[(-1)^n a_1 + a_2\left(q_{n,m} + 2p + 2\sum_{k=1}^{\infty}(-1)^k\eta_{k,m}\right)\right.$$
$$\left. + a_3(-1)^n\left(q_{n,m} + 4pn + 2\sum_{k=1}^{\infty}\eta_{k,m}\right)\right],$$

with a_1, a_2, a_3 constants and with $q_{n,m}$ defined in equation (12*b*). The recursion operator $\mathcal{L}_n$ is defined by [2]

$$\mathcal{L}_n \doteq -q_{n,m}\Delta_n^{(-)}\left(\Delta_n^{(+)}\right)^{-1}q_{n,m}E_n\left(\Delta_n^{(-)}\right)^{-1}\left(\Delta_n^{(+)}\right)^{-1}, \tag{39}$$

where $\Delta_n^{(\pm)} \doteq E_n \pm 1$. The inverses of $\Delta_n^{(\pm)}$ can be easily computed in the space of functions bounded at infinity and are given by

$$\left(\Delta_n^{(-)}\right)^{-1} = -\sum_{k=0}^{\infty} E_n^k, \qquad \left(\Delta_n^{(+)}\right)^{-1} = \sum_{k=0}^{\infty}(-1)^k E_n^k.$$

Choosing $V_n = 0$, from equation (38) a first equation is given by $\widetilde{V}_n = V_n^{(0)}$. Iterating this procedure we obtain the following set of equations:

$$\frac{\mathrm{d}\eta_{n,m}}{\mathrm{d}\varepsilon_k} = \mathcal{L}_n^k V_n^{(0)}, \qquad k \in \mathbb{N}. \tag{40}$$

Equations (40) involve always at least a summation and thus are not local. Following [2] we can obtain a set of local equations if we consider in equation (40) powers of the inverse of the recurrence operator $\mathcal{L}_n$. The explicit form of the inverse operator $\mathcal{L}_n^{-1}$ can be obtained from equation (39) and reads

$$\mathcal{L}_n^{-1} = -E_n^{-1}\Delta_n^{(-)}\Delta_n^{(+)}\frac{1}{q_{n,m}}\left(\Delta_n^{(-)}\right)^{-1}\Delta_n^{(+)}\frac{1}{q_{n,m}}.$$

Since $\Delta_n^{(-)}\alpha = 0$, where α is an arbitrary complex constant, we get that $\left(\Delta_n^{(-)}\right)^{-1}0 = \alpha$ (we can set $\alpha = 1$ without loss of generality). So applying $\mathcal{L}_n^{-1}$ to 0 we get a local formula for $\mathrm{d}\eta_{n,m}/\mathrm{d}\varepsilon_k$. By a recursive application we find

$$\frac{\mathrm{d}\eta_{n,m}}{\mathrm{d}\varepsilon_k} = \mathcal{L}_n^{-k-1}0. \tag{41}$$

Equations (41) turn out to be local and they form the so-called *inverse hierarchy*. Taking into account the form of $\mathcal{L}_n^{-1}$ we see that with any equation given in formula (41) we can associate an equation for the field $u_{n,m}$ given by

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\varepsilon_k} = \widetilde{\mathcal{L}}_n^{-k}\frac{1}{q_{n-1,m}} + \beta_k, \tag{42}$$

where

$$\widetilde{\mathcal{L}}_n^{-1} \doteq -E_n^{-1}\frac{1}{q_{n,m}}\Delta_n^{(+)}\left(\Delta_n^{(-)}\right)^{-1}\frac{1}{q_{n,m}}\Delta_n^{(+)}\Delta_n^{(-)}.$$

Here β_k's are some integration constants to be defined in such a way that $u_{n,m}$, asymptotically bounded, is a compatible solution of equation (42). In correspondence with equation (42) we obtain the following evolution of the reflection coefficient:

$$\frac{\mathrm{d}R_m}{\mathrm{d}\varepsilon_k} = \left[1 - \left(\frac{p+\mathrm{i}\kappa}{p-\mathrm{i}\kappa}\right)^k\right]R_m. \tag{43}$$

Taking into account equations (22) and (43) it follows that

$$E_m\frac{\mathrm{d}R_m}{\mathrm{d}\varepsilon_k} = \frac{\mathrm{d}R_{m+1}}{\mathrm{d}\varepsilon_k} \qquad \forall k \in \mathbb{N}, \tag{44}$$

where the m-shift operator is defined by $E_m^{\pm s}f_{n,m} \doteq f_{n,m\pm s}$, $s \in \mathbb{N}$, i.e. equation (42) is a symmetry of the lpKdV equation (2). Moreover, we also have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon_h}\left(\frac{\mathrm{d}R_m}{\mathrm{d}\varepsilon_k}\right) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon_k}\left(\frac{\mathrm{d}R_m}{\mathrm{d}\varepsilon_h}\right) \qquad \forall k, h \in \mathbb{N}, \tag{45}$$

i.e. the symmetries commute among themselves. As equation (33a) is a symmetry of the lpKdV equation (2) and the symmetries span a Lie algebra, also equation (42) with $\beta_k = 0$ is a symmetry. However, when $\beta_k = 0$ the spectral transform is not well defined.

In correspondence with any of the equations (42) with $\beta_k = 0$ we can construct an infinitesimal symmetry generator

$$\widehat{X}_n^{(k)} \doteq \widetilde{\mathcal{L}}_n^{-k} \frac{1}{q_{n-1,m}} \partial_{u_{n,m}}. \tag{46}$$

Thanks to equations (44), (45) the infinitesimal generators (46) satisfy the symmetry condition (24) and $\left[\widehat{X}_n^{(k)}, \widehat{X}_n^{(h)}\right] = 0 \; \forall k, h \in \mathbb{N}$.

For clarity of exposition let us write down the first three equations of the inverse hierarchy (for simplicity in terms of $q_{n,m}$ given by equation (11)).

For $k = 0$ we get

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\varepsilon_0} = -\frac{1}{q_{n-1,m}} + \frac{1}{2p}.$$

For $k = 1$ we have

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\varepsilon_1} = \frac{1}{q_{n-1,m}^2}\left(\frac{1}{q_{n,m}} + \frac{1}{q_{n-2,m}}\right) - \frac{1}{4p^3}.$$

For $k = 2$ we get

$$\begin{aligned}\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\varepsilon_2} = -\frac{1}{q_{n-1,m}^2}&\left[\frac{1}{q_{n-2,m}}\left(\frac{1}{q_{n,m}q_{n-1,m}} + \frac{1}{q_{n-1,m}q_{n-2,m}} + \frac{1}{q_{n-2,m}q_{n-3,m}}\right)\right.\\ &\left.+\frac{1}{q_{n,m}}\left(\frac{1}{q_{n+1,m}q_{n,m}} + \frac{1}{q_{n,m}q_{n-1,m}} + \frac{1}{q_{n-1,m}q_{n-2,m}}\right)\right] + \frac{3}{16p^5}.\end{aligned}$$

Before going over to the non-isospectral symmetries let us note that:

(1) As any of the previous equations can be rewritten just in terms of the variable q_n, the mappings $s_n \doteq (2p)/q_n$ and $a_n \doteq s_n s_{n-1}$ enable us to get the corresponding equations for the discrete KdV and Volterra hierarchies [7, 16].

(2) Due to the discrete symmetry $n \leftrightarrow m, p \leftrightarrow q$, there exists a symmetric isospectral hierarchy associated with the spectral problem (10) which give rise to a new denumerable class of commuting symmetries. These symmetries are given by equation (42) with $q_{n,m}$ substituted by

$$p_{n,m} \doteq 2q - u_{n,m+2} + u_{n,m},$$

and $\left(E_n^{\pm}, \Delta_n^{(\pm)}, \widetilde{\mathcal{L}}_n\right)$ substituted by $\left(E_m^{\pm}, \Delta_m^{(\pm)}, \widetilde{\mathcal{L}}_m\right)$. The infinitesimal symmetry generators of this hierarchy are given by

$$\widehat{X}_m^{(k)} \doteq \widetilde{\mathcal{L}}_m^{-k} \frac{1}{p_{n,m-1}} \partial_{u_{n,m}},$$

which are also commuting among themselves, i.e. $\left[\widehat{X}_m^{(k)}, \widehat{X}_m^{(h)}\right] = 0 \; \forall k, h \in \mathbb{N}$. Denoting by τ_k the continuous group parameter of this sequence, the first two symmetries of this class are thus given by

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\tau_0} = -\frac{1}{p_{n,m-1}} + \frac{1}{2q},$$

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\tau_1} = \frac{1}{p_{n,m-1}^2}\left(\frac{1}{p_{n,m}} + \frac{1}{p_{n,m-2}}\right) - \frac{1}{4q^3}.$$

(3) It is easy to prove that the two classes of symmetries we have constructed commute among themselves, i.e. $\left[\widehat{X}_n^{(k)}, \widehat{X}_m^{(h)}\right] = 0 \; \forall k, h \in \mathbb{N}$, when $\mathbb{D} = 0$.

4.2.2. The non-isospectral hierarchy. The non-isospectral hierarchy of equations is obtained by looking at those equations which are obtained from the Lax equation (37). Let us denote with σ (instead of ε) the continuous group parameter in this case.

The nonlinear differential-difference equations of the non-isospectral hierarchy are given by equation (38) with $V_n^{(0)} = q_n$. Note that $V_n^{(0)}$ is not asymptotically bounded, so that the resulting difference-differential equations obtained by applying the recursion operator $\mathcal{L}_n$ are not bounded in the limit $|n| \to \infty$. However, also in this case we can obtain asymptotically bounded equations by considering the inverse operator $\mathcal{L}_n^{-1}$. Taking into account that the solution of the equation $\Delta_n^{(-)}\alpha_n = \beta$ gives $\left(\Delta_n^{(-)}\right)^{-1}\beta = \beta n + \gamma$, where β and γ are constants with respect to the variation in n (but can depend on σ or m), we can construct a well-defined non-isospectral hierarchy of asymptotically bounded equations, starting from $V_n^{(0)}$ and choosing

$$f(\lambda) = 2\lambda^{-k}\left(\frac{\lambda}{p^2} + 1\right) \qquad k \in \mathbb{N}.$$

In this case we get

$$\frac{\mathrm{d}\eta_{n,m}}{\mathrm{d}\sigma_k} = \mathcal{L}_n^{-k}\left(\mathcal{L}_n^{-1} + \frac{1}{p^2}\right) q_{n,m}. \tag{47}$$

The first equation of the inverse hierarchy (47) is obtained for $k = 0$. The resulting equation is a non-autonomous deformation of equation (3). It reads

$$\frac{\mathrm{d}\eta_{n,m}}{\mathrm{d}\sigma_0} = \frac{2n-1}{q_{n-1,m}} - \frac{2n+3}{q_{n+1,m}} + \frac{q_{n,m}}{p^2},$$

which in terms of the field $u_{n,m}$ is given by

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\sigma_0} = \frac{2n - u_{n,m} - 1}{2p^2} + \frac{2n-1}{q_{n-1,m}}. \tag{48}$$

The higher order equations of the non-isospectral hierarchy (47) are all nonlocal. In correspondence with equation (48) we have the following evolution of the spectral data:

$$\frac{\mathrm{d}T_m}{\mathrm{d}\sigma_0} = 0, \qquad \frac{\mathrm{d}R_m}{\mathrm{d}\sigma_0} = -\frac{\mathrm{i}\kappa}{p(p^2+\kappa^2)} R_m, \tag{49}$$

with $\mathrm{d}\kappa/\mathrm{d}\sigma_0 = \kappa/p^2$.

We can now look for (n, m)-dependent symmetries working at the level of the spectrum, namely by looking at those evolutions of the reflection coefficients in correspondence with equations of the hierarchy (47) which commute with the discrete evolution of the reflection coefficient given by the lpKdV equation (2) and which thus satisfy equation (44) with ε_k substituted by σ_k. If we consider equation (48), as it is the only local one in the hierarchy (47) it is easy to see that

$$E_m \frac{\mathrm{d}R_m}{\mathrm{d}\sigma_0} \neq \frac{\mathrm{d}R_{m+1}}{\mathrm{d}\sigma_0}.$$

To get a symmetry we could add to equation (48) any equation of the isospectral hierarchy multiplied by an arbitrary function of m and κ, as it has been done in the case of the Toda Lattice [7]. To do so, let us add an unknown arbitrary function $\gamma_m(\kappa; p, q)$ to the evolution (49),

$$\frac{\mathrm{d}R_m}{\mathrm{d}\sigma_0} = \left(-\frac{\mathrm{i}\kappa}{p(p^2+\kappa^2)} + \gamma_m(\kappa; p, q)\right) R_m,$$

and let us require that $E_m(\mathrm{d}R_m/\mathrm{d}\sigma_0) = \mathrm{d}R_{m+1}/\mathrm{d}\sigma_0$ holds. In such a way we find

$$\gamma_m(\kappa; p, q) = -\frac{2\mathrm{i}\kappa m}{(q^2+\kappa^2)q}. \tag{50}$$

It is easy to prove that it is not possible to write the function $\gamma_m(\kappa; p, q)$ given by equation (50) as a finite combination of isospectral evolutions (43). So we are not able to construct in this way non-isospectral symmetries for the lpKdV equation.

If we are not interested in looking for asymptotic bounded equations we can extract from the non-isospectral hierarchy (47) the following local equations:

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\omega_0} = u_{n,m} - pn \qquad \text{for} \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\omega_0} = \frac{2}{p^2}, \tag{51a}$$

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}\omega_1} = \frac{n}{q_{n-1,m}} \qquad \text{for} \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\omega_1} = \frac{1}{\lambda}, \tag{51b}$$

where the group parameter is now denoted by ω. Equations (51) are not symmetries for the lpKdV equation (2). It is possible to show that equation (51b) is a *master symmetry* [6] of equation (2). In fact, let us define the following infinitesimal generators:

$$\widehat{Y}_n^{(1)} \doteq \frac{n}{q_{n-1,m}}\partial_{u_{n,m}}, \tag{52}$$

$$\widehat{Y}_m^{(1)} \doteq \frac{m}{p_{n,m-1}}\partial_{u_{n,m}}, \tag{53}$$

where $\widehat{Y}_m^{(1)}$ is obtained from $\widehat{Y}_n^{(1)}$ by applying the discrete symmetry $n \leftrightarrow m$, $p \leftrightarrow q$. As $\widehat{Y}_n^{(1)}$ (or equivalently $\widehat{Y}_m^{(1)}$) is not asymptotically bounded, one cannot associate with the corresponding equation (26) the evolution of a reflection coefficient and thus we cannot compute the commutation relations $\left[\widehat{X}_n^{(k)}, \widehat{Y}_n^{(1)}\right]$ for an arbitrary k. Let us write down the simplest commutation relations:

$$\left[\widehat{X}_n^{(0)}, \widehat{Y}_n^{(1)}\right] = -\widehat{X}_n^{(1)}, \qquad \left[\widehat{X}_n^{(1)}, \widehat{Y}_n^{(1)}\right] = -2\widehat{X}_n^{(2)}, \qquad \left[\widehat{X}_n^{(2)}, \widehat{Y}_n^{(1)}\right] = -\tfrac{3}{2}\widehat{X}_n^{(3)}.$$

A similar result could be obtained considering the commutation of $\widehat{X}_m^{(k)}$ and $\widehat{Y}_m^{(1)}$. As one can see from the above commutation relations the commutations of $\widehat{X}_n^{(k)}$ with $Y_n^{(1)}$ and of $\widehat{X}_m^{(k)}$ with $Y_m^{(1)}$ provide an alternative constructive tool with respect to the use of the recursive operator to find the infinite families of symmetries $\widehat{X}_n^{(k)}$, $\widehat{X}_m^{(k)}$ for the lpKdV equation (2).

4.2.3. Construction of non-autonomous generalized symmetries. We can now construct some new (n, m)-dependent non-autonomous symmetries of equation (2). Such a construction is provided by the following theorem whose proof is straightforward.

Theorem 1. *Let* $\mathbb{D}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; p, q) = 0$ *be an integrable partial difference equation invariant under the discrete symmetry* $n \leftrightarrow m$, $p \leftrightarrow q$. *Let* $\widehat{Z}_n$ *be the differential operator*

$$\widehat{Z}_n \doteq Z_n(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; p, q)\partial_{u_{n,m}},$$

such that

$$\mathrm{pr}\,\widehat{Z}_n\mathbb{D}|_{\mathbb{D}=0} = \alpha g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; p, q), \tag{54}$$

where $g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; p, q)$ *is invariant under the discrete symmetry* $n \leftrightarrow m$, $p \leftrightarrow q$ *and* α *is an arbitrary constant. Then*

$$\left(\frac{1}{\alpha}\,\mathrm{pr}\,\widehat{Z}_n - \frac{1}{\beta}\,\mathrm{pr}\,\widehat{Z}_m\right)\mathbb{D}\bigg|_{\mathbb{D}=0} = 0,$$

where the operator $\widehat{Z}_m \doteq Z_m(u_{n,m}, u_{n,m\pm1}, u_{n\pm1,m}, \ldots; q, p)\partial_{u_{n,m}}$ *is obtained from* $\widehat{Z}_n$ *under* $n \leftrightarrow m$, $p \leftrightarrow q$, *so that*

$$\mathrm{pr}\, \widehat{Z}_m \mathbb{D}|_{\mathbb{D}=0} = \beta g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; p, q),$$

being β *a constant. So*

$$\widehat{Z}_{n,m} \doteq \frac{1}{\alpha}\widehat{Z}_n - \frac{1}{\beta}\widehat{Z}_m \tag{55}$$

is a symmetry of $\mathbb{D} = 0$.

The above theorem provides a constructive tool to obtain generalized symmetries of the form given in equation (54) for the lpKdV equation (2). Such a situation occurs with the operators $\widehat{Y}_n^{(1)}$, $\widehat{Y}_m^{(1)}$ in equations (52) and (53) as in this case $\alpha g_{n,m} = -1$ and $\beta g_{n,m} = 1$. The symmetry $\widehat{Y}_n^{(1)} + \widehat{Y}_m^{(1)}$ has been considered by Tongas in [23] and the invariance condition one gets from it has been derived by Nijhoff and Papageorgiou [18] from a monodromy problem associated with equation (8*a*).

A more complicated symmetry for equation (2) can be obtained from theorem 1 by defining an operator $\widehat{Z}_n^{(w)}$, depending parametrically on $w \in \mathbb{R}$, with

$$Z_n^{(w)} \doteq \frac{np^w}{q_{n-1,m}} - \frac{p^w - q^w}{2(p^2 - q^2)}\left(pn - \frac{1}{2}u_{n,m}\right). \tag{56}$$

In such a case we find

$$\alpha g_{n,m} = -\frac{x_{n,m}^2(p^w - q^w) + 4pqx_{n,m}(p^{w-1} + q^{w-1}) + 4p^2q^2(p^{w-2} - q^{w-2})}{4(p+q)(x_{n,m} - p + q)},$$

where $x_{n,m} \doteq u_{n+1,m} - u_{n,m+1}$. The operator $\widehat{Z}_n \doteq Z_n^{(w)}\partial_{u_{n,m}}$, with $Z_n^{(w)}$ given by equation (56), satisfies the condition (54) and thus the operator (55) defines a one-parameter symmetry for the lpKdV equation (2). For $w = 0$ in equation (56) we recover the symmetry generated by $\widehat{Y}_n^{(1)} + \widehat{Y}_m^{(1)}$. The symmetry obtained by setting $w = 1$ in equation (56) can be also found in [23].

Let us give here two other examples of functions $Z_n^{(w)}$ satisfying condition (54):

$$Z_n^{(w)} \doteq \frac{n(p^w + q^w)}{q_{n-1,m}} - \frac{u_{n,m}}{p^w - q^w}, \tag{57}$$

$$Z_n^{(w)} \doteq \frac{n(pq)^w}{q_{n-1,m}} - \frac{u_{n,m}}{p^w - q^w}, \tag{58}$$

with $w \in \mathbb{R}\backslash\{0\}$.

Using theorem 1 we can also show that it is not possible to construct a symmetry using equation (51*a*). In fact, defining the differential operator

$$\widehat{Y}_n^{(0)} \doteq (u_{n,m} - pn)\partial_{u_{n,m}},$$

we immediately get that condition (54) is not satisfied as

$$\alpha g_{n,m} = \frac{qx_{n,m}^2 + 2p(p-q)x_{n,m} - 2p^2(p-q)}{x_{n,m} - p + q},$$

is not invariant under the exchange $n \leftrightarrow m$ and $p \leftrightarrow q$.

4.2.4. Symmetry reduction on the lattice. Let $\mathbb{D} = 0$ be a discrete equation and $\widehat{X}_{n,m} = F_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots)\partial_{u_{n,m}}$ one of its symmetry generators. A solution $u_{n,m}$ of $\mathbb{D} = 0$ is a *solution invariant* under $\widehat{X}_{n,m}$ if it satisfies the equation $\widehat{X}_{n,m}u_{n,m} = 0$.

We now use the notion of invariant solutions of the lpKdV equation (2) to construct a particular solution. Our results are a slight generalization of those contained in [19, 23].

Hereafter, carrying out the simplifying transformation

$$u_{n,m} \mapsto u_{n,m}\sqrt{p+q} + pn + qm, \tag{59}$$

we shall write the lpKdV equation (2) as

$$(u_{n+1,m} - u_{n,m+1})(u_{n+1,m+1} - u_{n,m}) = p - q. \tag{60}$$

Let us consider the symmetry generator $\widehat{X}_{n,m} \doteq \sqrt{p+q}\widehat{Z}^{(w)}_{n,m} + c\widehat{X}^{1}_{n,m}$, $c \in \mathbb{R}$, where $\widehat{Z}^{(w)}_{n,m} = \widehat{Z}^{(w)}_{n} + \widehat{Z}^{(w)}_{m}$, with $Z^{(w)}_{n}$ given in equation (56) and $\widehat{X}^{1}_{n,m}$ given in equation (33*a*). Under the map (59), the differential operator $\widehat{X}_{n,m}$ reads

$$\widehat{X}_{n,m} = \left(\frac{np^{w}}{u_{n+1,m} - u_{n-1,m}} + \frac{mq^{w}}{u_{n,m+1} - u_{n,m-1}} - \frac{1}{2}\frac{p^{w} - q^{w}}{p - q}u_{n,m} + c\right)\partial_{u_{n,m}}. \tag{61}$$

Solutions invariant under the symmetry generator (61) are solutions of equation (60) subject to the constraint

$$\frac{np^{w}}{u_{n+1,m} - u_{n-1,m}} + \frac{mq^{w}}{u_{n,m+1} - u_{n,m-1}} - \frac{1}{2}\frac{p^{w} - q^{w}}{p - q}u_{n,m} + c = 0. \tag{62}$$

Following [23] we define $a_{n,m} \doteq u_{n+1,m} - u_{n-1,m}$, $b_{n,m} \doteq u_{n,m+1} - u_{n,m-1}$ and $y_{n,m} \doteq u_{n+1,m+1} - u_{n,m}$. Using equation (60) we get

$$a_{n,m} = y_{n-1,m} + \frac{\delta}{y_{n,m}},$$

with $\delta \doteq p - q$. On the other hand, we have

$$\frac{1}{b_{n+1,m}} = \frac{\delta}{b_{n,m}y^{2}_{n,m}} + \frac{1}{y_{n,m}}. \tag{63}$$

Eliminating the variable $b_{n,m}$ between equations (62) and (63) we obtain the following second-order difference system:

$$\frac{p^{w}(n+1)\delta}{y_n y_{n+1} + \delta} + \frac{p^{w}n\delta}{y_n y_{n-1} + \delta} = p^{w}(n+1) + q^{w}m - c\frac{\delta}{y_n} + cy_n + \frac{p^{w} - q^{w}}{2\delta}\left(y_n u_n + \delta\frac{u_{n+1}}{y_n}\right), \tag{64a}$$

$$u_{n+1} - u_{n-1} = y_{n-1} + \frac{\delta}{y_n}, \tag{64b}$$

where we have dropped the index m in the dependent variables as it enters just parametrically. Equations (64*a*) and (64*b*) are a system extension of the alternate discrete Painlevé II equation [5, 19].

A similar procedure can be performed considering the operators (57) and (58). In these cases one obtains, as a reduction of equation (60), an equation that is equivalent to the alternate discrete Painlevé II equation for any value of w.

5. Concluding remarks

In this paper we have studied in detail the lpKdV equation and its symmetries. To do so in a complete way we started from its associated spectral problem. The study of the direct and inverse problem allow us to provide the soliton solutions of the nonlinear lattice equation. As has been shown in [17, 18] the associated spectral problem of the lpKdV equation is given by the asymmetric discrete Schrödinger spectral problem considered by Boiti *et al* [3]. A class of generalized symmetries of the lpKdV equation is obtained by considering its associated isospectral nonlinear evolution equations.

It is worthwhile to note here that the same transformation that exists between the Volterra equation (7) and the discrete KdV equation (6) can be written between the corresponding spectral problems. So part of the results we presented could be obtained from the discrete Schrödinger spectral problem associated with the Volterra equation (7). Equation (6) is associated with the spectral problem (5) which, in the variables of the discrete KdV equation, reads

$$\psi_{n+2} = \frac{2p}{s_n}\psi_{n+1} + \lambda\psi_n, \tag{65}$$

while the spectral problem for equation (7) is

$$\phi_{n-1} + a_n\phi_{n+1} = \mu\phi_n. \tag{66}$$

By up-shifting equation (66) by one, defining $a_n \doteq s_n s_{n-1}$ and considering the transformation $\phi_n \doteq f_n(\mu)g_n(s_n, s_{n\pm1}, \ldots)\psi_n$, we are able to rewrite equation (66) as equation (65). To do so we choose $f_n = \mu^n$ with $\lambda = -4p^2/\mu^2$ and g_n given by the solution of the recursion relation $g_{n+1} = g_n/(2ps_n)$. From this mapping, but in particular from the relation between λ and μ, we get that the positive Volterra hierarchy [7] corresponds to a negative hierarchy in the asymmetric discrete Schrödinger spectral problem (65). In this way the Miura transformation between the potential $a_{n,m}$ of the Volterra hierarchy and field $u_{n,m}$ of the lpKdV equation is easily obtained and is given by

$$a_{n,m} = \frac{4p^2}{(2p - u_{n+2,m} + u_{n,m})(2p - u_{n+1,m} + u_{n-1,m})}.$$

From the two equivalent components of the Lax pair we get two infinite hierarchies of integrable symmetries by looking at its isospectral deformations. From the non-isospectral deformations we get an infinite set of master symmetries. Apart from the two classes of integrable symmetries, associated with the two equivalent members of the Lax pair, we have obtained an infinite set of symmetries starting from the master symmetries. It is an open problem to prove whether the symmetries $\widehat{Z}^{(w)}$ are integrable, possibly by providing their Lax pair and to compute their commutation relations. In any case we can use them to provide solutions by symmetry reducing the lpKdV equation.

In conclusion we have shown that the lpKdV equation has a lot of symmetries. It is left to future work the proof if all are independent or if, the two sequences associated with the spectral problems in n and m respectively are dependent under the lpKdV equation.

It is also to be clarified the role of the $\widehat{Z}^{(w)}$ symmetries, related to the master symmetry. It is an open problem to see whether the other discrete symmetries might give rise to new continuous symmetries. We leave to a future work the use of the $\widehat{Z}^{(w)}$ symmetries here obtained to check the integrability of the (local) dNLS equations derived in [9, 10, 12].

After we finished this paper a referee for our recent paper [13] pointed out to us the article by Rasin and Hydon [21] submitted to *Stud. Appl. Math.* and available in the personal web page of the authors. In this article the authors construct the symmetries for all quad-graph equations contained in the Adler–Bobenko–Suris list [1]. Let us note here that in the present paper we have just considered the H_1 equation but with much more details. In particular we already answer in this case to some of the questions in their conclusions. Moreover we have obtained five point symmetries that they have not been able to find.

Acknowledgments

The authors were partially supported by PRIN Project 'SINTESI-2004' of the Italian Minister for Education and Scientific Research. This research is part of a joint Italian Russian research project 'Classification of integrable discrete and continuous models' financed by Consortium EINSTEIN and Russian Foundation for Basic Research. The authors thank C Scimiterna for some fruitful discussions.

References

[1] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513–43

[2] Boiti M, Bruschi M, Pempinelli F and Prinari B 2003 A discrete schrödinger spectral problem and associated evolutions equations *J. Phys. A: Math. Gen.* **36** 139–49

[3] Boiti M, Pempinelli F, Prinari B and Spire A 2001 An integrable discretization of kdV at large times *Inverse Problems* **17** 515–26

[4] Bruschi M and Ragnisco O 1980 Extension of the lax method to solve a class of nonlinear evolution equations with x-dependent coefficients associated to the matrix Schrödinger spectral problem *Lett. Nuovo Cimento* **29** 327–30

[5] Fokas A S, Grammaticos. B and Ramani A 1993 From continuous to discrete painlevé equations *J. Math. Anal. Appl.* **180** 342–60

[6] Fuchssteiner B and Ma W-X 1999 An approach to master symmetries of lattice equations *Symmetries and Integrability of Difference Equations (Canterbury, 1996) (London Math. Soc. Lecture Note Ser.* vol 255*)* (Cambridge: Cambridge University Press) pp 247–60

[7] Heredero R H, Levi D, Rodriguez M A and Winternitz P 2000 Lie algebra contractions and symmetries of the Toda hierarchy *J. Phys. A: Math. Gen.* **33** 5025–40

[8] Hirota R 1977 Nonlinear partial difference equations I. A difference analogue of the Korteweg-de Vries equation *Jour. Phys. Soc. Japan* **43** 1424–33

[9] Ji, Xiaoda, Levi D and Petrera M Multiscale expansion of the lattice sine-Gordon equation *Proceedings of 'International Conference on Difference Equations, Special Functions and Applications' (25–30 July 2005, Munich, Germany)* www.arxiv.org/math-ph/0511006

[10] Levi D 2005 Multiple-scale analysis of discrete nonlinear partial difference equations: the reduction of the lattice potential kdV *J. Phys. A: Math. Gen.* **38** 7677–89

[11] Levi D and Martina L 2001 Integrable hierarchies of nonlinear difference-difference equations and symmetries *J. Phys. A: Math. Gen.* **34** 10357–68

[12] Levi D and Petrera M 2006 Discrete reductive perturbation technique *J. Math. Phys.* **47** 543–63

[13] Levi D, Petrera M and Scimiterna C The lattice Schwarzian KdV equation and its symmetries *J. Phys. A: Math. Gen.* (special issue SIDE VII) submitted http://www.arxiv.org/pdf/math-ph/0701044

[14] Levi D and Winternitz P 2006 Continuous symmetries of difference equations *J. Phys. A: Math. Gen.* **39** R1–R63

[15] Mikhailov A V, Shabat A B and Yamilov R I 1987 A symmetric approach to the classification of nonlinear equations. Complete lists of integrable systems *Uspekhi Mat. Nauk* **42** 3–53

[16] Narita K 1998 Solutions for the mikhailov-shabat-Yamilov difference-differential equations and generalized solutions for the volterra and the toda lattice equations *Prog. Theor. Phys.* **99** 337–48

[17] Nijhoff F W and Capel H W 1995 The discrete Korteweg-de Vries equation *Acta Appl. Math.* **39** 133–58

[18] Nijhoff F W and Papageorgiou V G 1991 Similarity reductions on integrable lattices and discrete analogues of the Painlevé II equation *Phys. Lett.* A **153** 337–44

[19] Nijhoff F W, Sastuma J, Kajiwara K, Grammaticos B and Ramani A 1996 A study of the alternate discrete painlevé equation *Inverse Problems* **12** 697–716

[20] Olver P 1995 *Equivalence, invariants and symmetry* (Cambridge: Cambridge University Press)

[21] Rasin O G and Hydon P E Symmetries of integrable difference equations on the quad-graph *Stud. Appl. Math* submitted

[22] Taniuti T and Nishihara K 1983 Nonlinear waves *Monographs and Studies in Mathematics* (Boston: Pitman)

[23] Tongas A 2005 On the symmetries of integrable partial difference equations *Proceedings of 'International Conference on Difference Equations, Special Functions and Applications' (25–30 July 2005, Munich, Germany)*

[24] Yamilov R 2006 Symmetries as integrability criteria for differential difference equations *J. Phys. A: Math. Gen.* **39** R541–R623